Unsteady Flow in a Supersonic Cascade with Subsonic Leading-Edge Locus

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Linearized theory is used to predict the unsteady flow in a supersonic cascade with subsonic axial flow velocity. A closed-form analytical solution is obtained by using a double application of the Wiener-Hopf technique. Although numerical and semianalytical solutions of this problem have already appeared in the literature, this paper contains the first completely analytical solution. It has been stated in the literature that the blade source should vanish at the infinite duct resonance condition. The present analysis shows that this does not occur. This apparent discrepancy is explained in the paper.

I. Introduction

ONE of the most serious problems encountered in the development of modern turbojet engines is the blade flutter in the fan and compressor stages. This often occurs when the blades are unstalled and operating supersonically. It is therefore important to be able to predict the occurrence of this phenomenon mathematically. The present analysis is directed toward this goal.

As is usually done, we represent an incremental annulus of the compressor or fan stage by a rectilinear two-dimensional cascade. Modern supersonic fans and compressors operate with flow velocities which are supersonic relative to the blades with with subsonic axial velocities entering the blade row. This is the so called "subonic leading-edge locus problem."

In order to predict blade flutter we allow the blades to undergo a small-amplitude harmonic oscillation which superposes a small unsteady flow on the existing steady flowfield. Since the blades are thin, it is usual to assume that the steady flow deviates only slightly from a uniform flow. Hence both the steady and unsteady flows are treated as small perturbations about a uniform "basic" flow. A number of analyses both semianalytical[1-5] and numerical[6,7] have been performed for the case where the basic flow is supersonic. However, none of these papers give an exact closed-form solution to this problem. This is done in the present report.

As is well known, the unsteady and steady-state aerodynamics decouple in the linearized approximation so that the unsteady flow can be calculated independently of the steady flow perturbations. Indeed, if we are only interested in the unsteady blade forces, we need never consider the steady flow perturbations. Moreover, the thickness, camber, and mean angle of attack of the blades can be shown to influence only the steady flow perturbations so that for the purposes of calculating the unsteady flow we can replace the blades by a set of zero thickness flat plates which are shown in Fig. 1. The fluid is assumed to be an inviscid nonheat-conducting ideal gas with constant specific heats and the flow is assumed to be irrotational and isentropic.

In Sec. II the problem is formulated and the governing equations and boundary conditions are deduced. Since disturbances cannot propagate upstream in a supersonic flow, it is easy to see from Fig. 1 that the flow downstream of the Mach wave emanating from the trailing edge cannot influence the flow upstream of this line. We take advantage of this fact

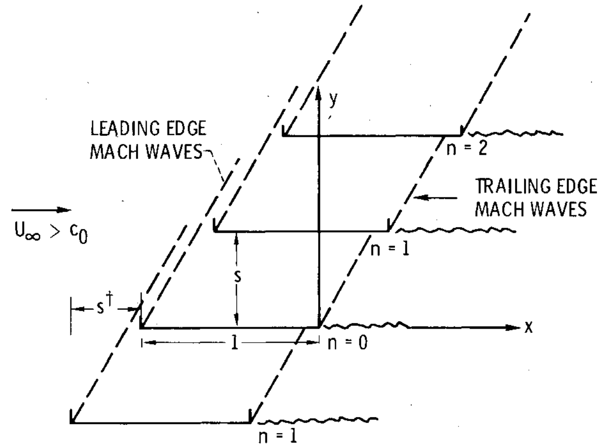

Fig. 1 Cascade configuration.

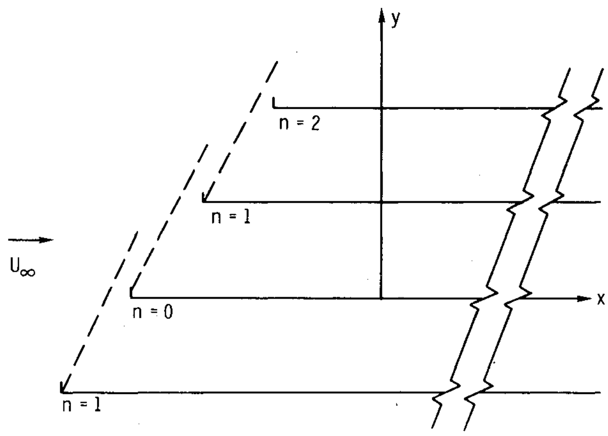

Fig. 2 Configuration for upstream solution.

by calculating the flow upstream of these Mach waves by replacing the blades by a row of semi-infinite flat plates as shown in Fig. 2. The analytical solution to this problem is developed by Goldstein et al.[8] by employing the Wiener-Hopf technique. (This technique is discussed in a number of texts, e.g., Noble.[9]) The solution downstream of the trailing-edge Mach wave is obtained by considering the backward facing row of semi-infinite plates shown in Fig. 3. In this problem we assume that the normal velocity vanishes on the plates and that the jump in pressure across the wakes is equal and opposite to that given by the upstream solution. The analytical solution is again obtained by the Wiener-Hopf technique. It

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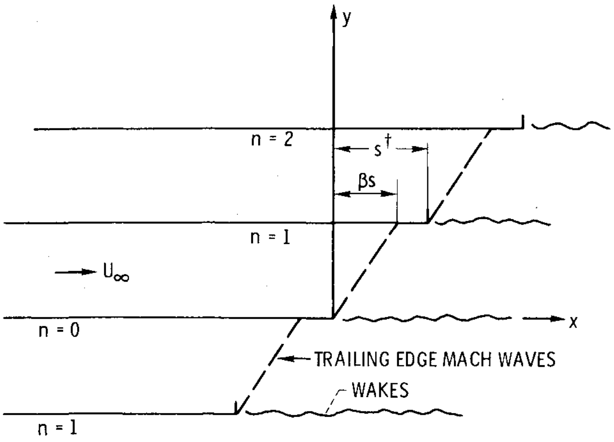

Fig. 3 Configuration for downstream solution.

turns out that this solution is identically zero upstream of the trailing Mach waves. Hence, when it is added to the upstream solution, an exact closed-form solution that satisfies all of the boundary conditions is obtained.

The analysis is analogous to but differs in a nontrivial way from the one used by Mani and Horvey.[10] But, since the flow is subsonic in their analysis, their solution turns out to be only approximate while the present solution is exact. Numerical results are presented in Sec. IV, and the implications of the solution about supersonic flutter are discussed.

II. Formulation

We suppose that all lengths are nondimensionalized by the blade chord c; the time t is nondimensionalized with respect to c divided by the undisturbed freestream velocity U_∞. The pressure fluctuation p' is nondimensionalized by the undisturbed freestream density ρ_0 multiplied by U_∞^2, and all fluctuating velocities are nondimensionlized by U_∞.

As already indicated, the blades can be replaced by flat plates oscillating harmonically about their mean positions $y=ns$, $ns^\dagger - 1 < x < ns^\dagger$, for $n=0, \pm 1, \pm 2, \ldots$.

To this order of approximation we can assume that the unsteady wakes are vortex sheets emanating from the trailing edge of the blades and having mean positions along the lines $y=ns$, $x>s^\dagger n$ for $n=0, \pm 1, \pm 2, \ldots$. As shown by Lane,[11] it is only necessary for the purpose of studying flutter to consider blade motions in which all blades oscillate harmonically with the same amplitude and a constant but arbitrary interblade phase angle σ. Since the problem is linear, all motion induced by the harmonic oscillations of the blades must also have harmonic time dependence. This motion is determined by the velocity potential

$$\phi(x,y,t) = \Phi(x,y) e^{-i\omega_l t} \tag{1}$$

where $\omega_l = \omega c/U_\infty$ is the reduced frequency, and Φ is governed by the moving medium reduced wave equation

$$\left(\frac{\partial^2}{\partial y^2} - \beta^2 \frac{\partial^2}{\partial x^2} + 2iM\beta^2 k \frac{\partial}{\partial x} + \beta^4 k^2\right)\Phi = 0 \tag{2}$$

where $M = U_\infty / a_0 > 1$ is the undisturbed freestream Mach number (based on the undisturbed freestream speed of sound a_0),

$$\beta = \sqrt{M^2 - 1} \tag{3}$$

and

$$k \equiv \omega_l M/\beta^2 \tag{4}$$

Once the velocity potential amplitude Φ is known, the amplitudes

$$P = p' \exp[i\omega_l t] \quad V = v \exp[i\omega_l t] \quad U = u \exp[i\omega_l t] \tag{5}$$

of the pressure fluctation p' and the upwash and axial velocity fluctuations v and u, respectively, can be determined by differentiation from the relations

$$V = \frac{\partial \Phi}{\partial y}, \quad U = \frac{\partial \Phi}{\partial x} \tag{6}$$

and

$$P = \left(i\omega_l - \frac{\partial}{\partial x}\right)\Phi \tag{7}$$

Since the upwash velocity on the nth blade is assumed to differ from that on the zeroth blade only by a constant phase factor σ, it follows that

$$V(x+ns^\dagger, ns \pm 0) = e^{in\sigma} V(x, \pm 0) \quad \text{for} \quad -1 < x < 0$$
$$n = 0, \pm 1, \pm 2, \ldots \tag{8}$$

where $+0$ denotes the limit as $y \to 0$ from above while -0 denotes the limit as $y \to 0$ through negative values. This equation determines the upwash velocity on the nth blade in terms of that of the zeroth blade while the upwash velocity on the zeroth blade is related to its displacement $W_0 e^{-i\omega_l t}$ by

$$V(x, \pm 0) = -\left(i\omega_l - \frac{\partial}{\partial x}\right) W_0(x) \quad \text{for } y = 0, -1 < x < 0$$

The blade can, in general, be undergoing any type of undulation. But in order to simplify the presentation we shall restrict our attention to the case usually considered in compressor flutter calculations wherein each incremental blade section is undergoing a rigid body motion. Then we can write

$$W_0 \equiv H_0 + A_0(x - d_0) \tag{9}$$

where H_0, A_0, and d_0 are constants. H_0 represents the amplitude of a vertical displacement of the point $x = d_0$ while A_0 is the amplitude of the angular displacement about this point.

Across the wake we require that the pressure and upwash velocity be continuous. Finally, we require that far downstream all disturbances travel away from the cascade. In the region ahead of the cascade we require that only outward propagating waves exist. This completes the specification of the problem. We now proceed to construct a solution which satisfies these conditions.

III. Analytical Solution

Before constructing this solution we shall assume, as is usually done, that there is a small amount of damping in the problem. This amounts to requiring that k [which appears in Eq. (2)] has a small positive imaginary part, say ϵ_l. At the end of the problem we set the damping equal to zero. This allows us to replace the outgoing wave boundary conditions at infinity by conditions of boundedness.

The boundary condition (8) requires that the solution possess a certain blade-to-blade periodicity. We shall require that the solution actually satisfy the stronger periodicity condition

$$\theta(x + ns^\dagger, ns + y) - e^{in\sigma}\theta(x,y) \tag{10}$$

where θ can denote any of the physical variables V, P, or U. Of course, once we demonstrate that such a solution can be made to satisfy all of the boundary conditions in the problem we can conclude that this assumption was indeed justified.

Since, as indicated in Sec. I and as pointed out in Ref. 8, disturbances cannot propagate upstream in a supersonic flow, it is easy to see from Fig. 1 that any disturbances originating at or behind the Mach waves emanating from the trailing edges have no influence on the flow in the region upstream of these lines. Hence, the flow in this region can be calcualted independently of the solution in the downstream region. We can therefore suppose (as was done in Ref. 8) that for purposes of calculating the upstream solution, the blades can be extended downstream to infinity. The cascade can therefore be treated as a row of semi-infinite flat plates.

Let Φ_1 denote the solution to this problem. Then $\Phi=\Phi_1$ upstream of the trailing Mach waves and downstream of these lines there is a function Φ_2 such that $\Phi=\Phi_1+\Phi_2$. The solution Φ_1 was obtained by Goldstein et al. in Ref. 8. It is given by

$$\Phi_1=\frac{1}{2\pi}\int_{-\infty+i\epsilon_1 M}^{\infty+i\epsilon_1 M}\exp[-i(\alpha-Mk)(x+1)]\frac{\kappa^-(MK-i\epsilon_0)}{\kappa^+(\alpha)}$$

$$\times\frac{\Lambda(\alpha,y)}{(\alpha-Mk+i\epsilon_0)}\left(D_-+\frac{i\omega_1 A_0}{\alpha-Mk+i\epsilon_0}\right)\mathrm{d}\alpha \quad \text{for } 0\le y\le s \tag{11}$$

where

$$\Lambda(\alpha,y)=\frac{1}{2i}\left\{\frac{\exp[i(\Delta^-+\beta\gamma y)]}{\sin\Delta^-}+\frac{\exp[i(\Delta^+-\beta\gamma y)]}{\sin\Delta^+}\right\} \tag{12}$$

$$\Delta^{\pm}\equiv ½(\sigma-Mks^\dagger+\alpha s^\dagger\pm\beta\gamma s) \tag{13}$$

$$\kappa(\alpha,y)\equiv\frac{\partial}{\partial y}\Lambda(\alpha,y)$$

$$\gamma\equiv\sqrt{\alpha^2-k^2}$$

$$D_-=iA_0\left\{1+\omega_1\frac{[\kappa^-(Mk-i\epsilon_0)]'}{\kappa^-(Mk-i\epsilon_0)}\right\}+\omega_1[H_0-A_0(1+d_0)]$$

and in order to insure that no waves propagate upstream and that the solution remain bounded at infinity (i.e., only outward propagating waves exist) we have chosen the branch cut for the square root γ and the integration contour in the complex α plane in the manner indicated in Fig. 4. The quantity ϵ_0 is a small positive number which can be put equal to zero after the contour integral is evaluated and $\kappa^{\pm}(\alpha)$ are determined by

$$\kappa^-(\alpha)=\exp\left[-i\left(\frac{s^\dagger-\beta s}{2}\right)\alpha\right]\prod_{n=-\infty}^{\infty}\left(1-\frac{\alpha}{\nu_n^+}\right)$$

$$\kappa(\alpha,0)=\frac{\beta\gamma\sin\beta\gamma s}{2\sin\Delta^+\sin\Delta^-}$$

and

$$\kappa(\alpha,0)=\frac{\kappa^+(\alpha)}{\kappa^-(\alpha)} \quad \operatorname{Im}\ \alpha=\epsilon_1 M$$

where

$$d^\dagger=\sqrt{s^{\dagger 2}-\beta^2 s^2}$$

$$\nu_n^{\pm}=\Gamma_n\frac{s^\dagger}{d^\dagger}\pm\frac{s\beta}{d^\dagger}\sqrt{[\Gamma_n]^2-k^2} \quad \text{for } n=0,\pm 1,\pm 2,\ldots$$

$$\Gamma_n\equiv\frac{2n\pi+Mks^\dagger-\sigma}{d^\dagger}$$

and the branch cut for the square root is the one shown in Fig. 4. It is also shown in Ref. 8 as $\alpha\to\infty$

$$\kappa^-(\alpha)\sim\kappa_\infty[1-\exp\{-i[\alpha(s^\dagger-\beta s)+\sigma-Mks^\dagger]\}]$$

where

$$\kappa_\infty\equiv\frac{-i\exp[(i/2)(\sigma-Mks^\dagger)]}{2\sin[(\sigma-Mks^\dagger)/2]}\prod_{n=-\infty}^{\infty}\frac{\Gamma_n d^\dagger}{\nu_n^+(s^\dagger-\beta s)}$$

The equation for the surface pressure is obtained by substituting Eq. (11) into Eq. (7) and carrying out the indicated differentiation. But as shown in Ref. 8, the singularities in the integrand of the resulting expression for the airfoil surface pressure are at most poles. Hence the integrals can be evaluated by the method of residues. On the portion of the surface $y=0$ and $-1<x<-1+s^\dagger-s\beta$ the integration contour must be closed on a large semicircle in the upper half-plane while on the remaining portions, $y=0$, $-1+s^\dagger-s\beta<x<0$ and $y=s$, $-1+s^\dagger<x<s^\dagger$ it must be closed in the lower half-plane. These operations have been carried out in Ref. 8 for the surface at $y=0$ (i.e., the upper surface of the blade) where it is shown that

$$P(x,0)=P_1(x,0)=-is\sum_{n=-\infty}^{\infty}\frac{[\nu_n^+-(k/M)]\kappa^-(Mk)}{[\nu_n^+-Mk][\kappa^-(\nu_n^+)]'}\,\frac{\exp[-i(\nu_n^+-Mk)(x+1)]}{d^\dagger\Gamma_n-\nu_n^+s^\dagger}\left(D_-+\frac{i\omega_1 A_0}{\nu_n^+-Mk}\right)$$

$$\text{for } -1<x<-1+s^\dagger-s\beta \tag{14}$$

where $[\kappa^-(x)]'\equiv(\mathrm{d}/\mathrm{d}x)\kappa^-(x)$ and the subscript 1 is used to indicate that the pressure is obtained from Φ_1. Similarly,

$$P(x,0)=P_1(x,0)=\exp[iMk(x+1-s^\dagger)]\sum_n\left[T_n^+\exp[-i\lambda_n(x+1-s^\dagger)]+T_n^-\exp[i\lambda_n(x+1-s_2^\dagger)]\right]$$

$$-\omega_1^2[H_0+A_0(x-d_0)]f_1(0,\omega_1)-iA_0\left[\frac{\partial}{\partial\omega_1}\omega_1^2 f_1(0,\omega_1)+\omega_1^2 f_2(0,\omega_1)\right] \quad \text{for } -1+s^\dagger-s\beta<x<0 \tag{15}$$

where

$$T_n^{\pm}\equiv-\left[\frac{\pm\lambda_n-(k/M)}{\pm\lambda_n-Mk}\right]Q_n^{\pm}, \quad \lambda_n=\sqrt{k^2+\frac{n^2\pi^2}{\beta^2 s^2}} \quad \text{for } n=0,\pm 1,\pm 2,\ldots, \quad Q_n=Q_n^++Q_n^-$$

$$Q_n^{\pm}=\frac{\exp[i(\sigma-n\pi)]-\exp[i(Mk\mp\lambda_n)s^\dagger]}{\pm\lambda_n s\beta^2(1+\delta_{n,0})}\,\frac{\kappa^-(Mk)}{\kappa^-(\pm\lambda_n)}\left(D_-+\frac{i\omega_1 A_0}{\pm\lambda_n-Mk}\right)$$

$$f_1(y;\omega)\equiv\frac{e^{i\sigma}\cos M\omega y-\cos M\omega(s-y)}{M\omega\sin M\omega s}, \quad f_2(y;\omega)\equiv\frac{is^\dagger e^{i\sigma}\cos(M\omega y)}{M\omega\sin(M\omega s)}$$

Similar manipulations can be used to evaluate the pressure on the lower portion of the $n=1$ surface and thereby obtain

$$P_1(x,s)=\sum_{n=0}^{\infty}(-1)^n\{T_n^+\exp[i(Mk-\lambda_n)(x+1-s^\dagger)]+T_n^-\exp[i(Mk+\lambda_n)(x+1-s^\dagger)]\}$$

$$-\omega_1^2[H_0+A_0(x-d_0)]f_1(s,\omega_1)-iA_0\left[\frac{\partial}{\partial\omega_1}\omega_1^2 f_1(s,\omega_1)+\omega_1^2 f_2(s,\omega_1)\right]\quad\text{for}\quad -1+s^\dagger<x<s^\dagger \tag{16}$$

Now since, as we have indicated, Φ_1 does not represent the solution to the problem in the region downstream of the trailing-edge Mach wave, it must be augmented by a solution Φ_2. We choose Φ_2 to be a solution to Eq. (2) that satisfies the boundary conditions (see Fig. 3)

$$V_2(x+ns^\dagger,ns)=0\quad\text{for}\quad -\infty<x<0 \tag{17}$$

and

$$[P_2(x)]_n=-[P_1(x)]_n\quad\text{for}\quad 0<x<\infty;n=0,\pm1,\pm2,\ldots \tag{18}$$

where $[P(x)]_n$ denotes the jump $P(x+ns^\dagger,ns+0)-P(x+ns^\dagger,ns-0)$. Then $\Phi=\Phi_1+\Phi_2$ will satisfy the correct boundary conditions on the surface of the blades and across the wake. However, it will not necessarily satisfy the correct conditions across the horizontal lines extending to upstream infinity from the leading edges of the blades. On the other hand, these conditions will certainly be satisfied if Φ_2 is identically zero in the region upstream of the trailing-edge Mach waves. We shall show that this is the case once the solution is constructed. Thus, $\Phi_1+\Phi_2$ will be the exact solution to the problem.

A similar procedure was used by Mani and Horvay[10] in connection with the transmission of sound through a blade row. But since their analysis was for subsonic flow the solution Φ_2 did not vanish upstream and the calculation was therefore approximate.

In order to determine Φ_2 we must first calculate $[P_1(x)]_0$ across the wake. This can be done by substituting Eq. (11) into Eq. (7) and evaluating the contour integrals in the manner described in conjunction with the surface pressure calculations. The result is

$$[P_1(x)]_0=\sum_{n=0}^{\infty}\left\{S_n^+\exp[i(Mk-\lambda_n)x]+S_n^-\exp[i(Mk+\lambda_n)x]\right\}$$

$$+\tilde{B}_1+\tilde{B}_2 x \tag{19}$$

where

$$S_n^\pm\equiv\pm2\left(\pm\lambda_n-\frac{k}{M}\right)\frac{1-(-1)^n\cos(\sigma-Mks^\dagger\pm\lambda_n s^\dagger)}{s\beta^2\lambda_n(1+\delta_{n,0})(\pm\lambda_n-Mk)}$$

$$\times\frac{\kappa^-(Mk)}{\kappa^-(\pm\lambda_n)}\exp[i(Mk\mp\lambda_n)]\left(D_-+\frac{i\omega_1 A_0}{\pm\lambda_n-Mk}\right)$$

$$\tilde{B}_1=\frac{2\omega_1}{\kappa(Mk)}\left\{iA_0\left[2-\omega_1\frac{\kappa'(Mk)}{\kappa(Mk)}\right]+\omega_1(H_0-d_0A_0)\right\}$$

$$\tilde{B}_2=\frac{2\omega_1^2 A_0}{\kappa(Mk)}$$

$$\kappa(\alpha)\equiv\kappa(\alpha,0)\qquad\kappa'(Mk)\equiv\left.\frac{\mathrm{d}\kappa(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=Mk}$$

The form of Eq. (19) and the boundary condition (18) suggest that we seek a solution of the form

$$\Phi_2=\sum_{n=0}^{\infty}\Phi_\pm^{(n)}+\tilde{\Phi}_1+\tilde{\Phi}_2 \tag{20}$$

where (as explained in Ref. 8) in order to insure that each Φ is an outgoing wave solution to Eq. (2) that satisfies the periodicity condition of Eq. (10) we must take

$$\Phi_\pm^{(n)}=\frac{1}{2\pi}\int_{-\infty+i\epsilon_1 M}^{\infty+i\epsilon_1 M}f_n^\pm(\alpha)\Lambda(\alpha,y)\exp[-i(\alpha-Mk)x]\mathrm{d}\alpha \tag{21}$$

$$\tilde{\Phi}_j=\frac{1}{2\pi}\int_{-\infty+i\epsilon_1 M}^{\infty+i\epsilon_1 M}\tilde{f}_j(\alpha)\Lambda(\alpha,y)\exp[-i(\alpha-Mk)x]\mathrm{d}\alpha,$$

$$\text{for}\quad j=1,2 \tag{22}$$

Then we need only impose the boundary conditions along the line $y=0$ (i.e., for $n=0$) and they will automatically be satisfied along the remaining lines $y=ns$ for $n=\pm1,\ \pm2,\ldots$. The $\Phi_\pm^{(n)}$ and $\tilde{\Phi}_j$ must therefore satisfy the conditions

$$V_\pm^{(n)}=\tilde{V}_j=0;\quad x<0,y=0$$

and

$$[P_\pm^{(n)}]=-S_n^\pm\exp[i(Mk\mp\lambda_n)x],\ \ [\tilde{P}_j]=-\tilde{B}_j x^{-(1-j)}$$

$$j=1,2\quad\text{for}\quad x>0,y=0$$

Substituting Eqs. (21) and (22) via Eqs. (6) and (7) into these conditions leads to sets of dual integral equations for the quantities $f_n^\pm,\tilde{f}_j$ that can be solved by employing the Wiener-Hopf procedure in precisely the same manner as was done in Ref. 8 to determine Φ_1. The solutions are

$$f_n^\pm=\frac{S_n^\pm}{2}\frac{\kappa^-(\alpha)}{\kappa^-(\pm\lambda_n)}\frac{1}{\left(\dfrac{k}{M}-\alpha\right)(\alpha\mp\lambda_n)} \tag{23a}$$

$$\tilde{f}_1=\frac{\tilde{B}_1}{2}\frac{\kappa^-(\alpha)}{\kappa^-(Mk)}\frac{1}{\left(\dfrac{k}{M}-\alpha\right)(\alpha-Mk+i\epsilon_0)} \tag{23b}$$

$$\tilde{f}_2=\frac{i\tilde{B}_2\kappa^-(\alpha)}{2\kappa^-(Mk)\left(\dfrac{k}{M}-\alpha\right)(\alpha-Mk+i\epsilon_0)}$$

$$\times\frac{1}{\alpha-Mk+i\epsilon_0}-\frac{[\kappa^-(Mk)]'}{\kappa^-(Mk)} \tag{23c}$$

These results can now be substituted into Eqs. (21) and (22), and as explained in conjunction with Φ_1, the method of residues can be used to evaluate the contour integrals. For

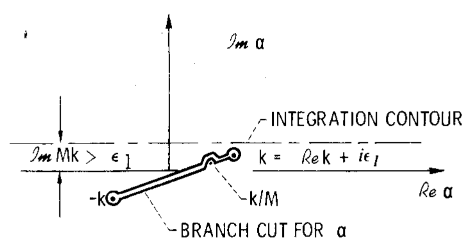

Fig. 4 Integration contour and branch cut in complex α plane.

Table 1 Verdon's cascade A

Interblade phase angle, deg	Moment coefficient ratio, C_m/A_0			
	Present analysis		Verdon's analysis	
	Real	Im	Real	Im
−180	0.5212	−0.3641	0.51	−0.37
−150	−0.3309	−0.5030	−0.34	−0.5
−120	0.08566	−0.5153	0.11	−0.57
−90	−0.1790	−0.3910	−0.175	−0.408
−85.384	−0.4694	−0.8881	Resonance	
−60	0.1617	−0.2895	0.160	−0.290
−30	−0.1872	−0.2824	−0.190	−0.28
0	−0.2354	0.05372	−0.24	0.04
30	−0.05435	0.2602	−0.04	0.26
60	0.1851	0.3257	0.19	0.30
90	0.4123	0.2567	0.405	0.24
120	0.5674	0.08128	0.58	0.05
150	0.6073	−0.1479	0.60	−0.16
180	0.5212	−0.3641	0.51	−0.37

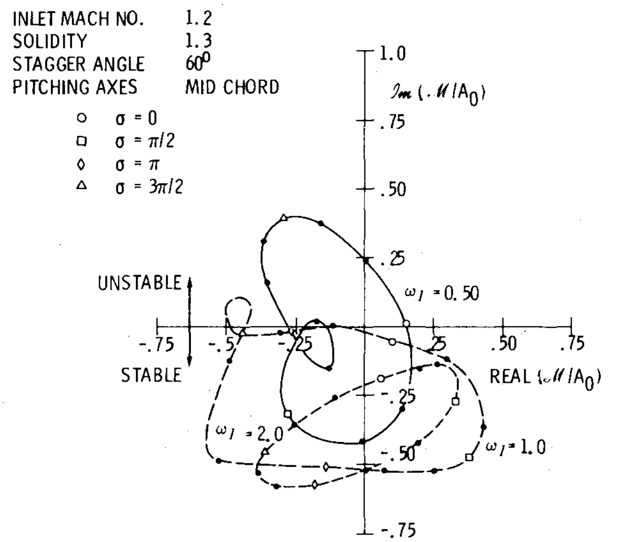

Fig. 5 Complex moment coefficient for pitching motion about center of airfoil. Black dots denote intermediate values (i.e., multiples of π/σ) of interblade phase angle σ. Mach number 1.2.

points in the region upstream of the Mach wave emanating from the trailing edge, the contour must be closed in the upper half-plane. But since all of the poles lie in the lower half-plane, $\Phi_\pm^{(n)}$ and $\tilde\Phi_j$ will be identically zero in this region. In view of Eq. (20), this proves our assertion that Φ_2 is identically zero in this region and therefore that $\Phi=\Phi_I+\Phi_2$ is the correct solution. The quantity of interest in the region downstream of the trailing-edge Mach wave is the pressure on the lower surface $y=s, 0<x<s^\dagger$. This can be calculated by substituting Eqs. (23) into (21) and (22). The resulting integral expressions are then introduced into Eq. (20) whereupon the result is substituted into Eq. (7). The integral expression for P_2 can be evaluated by the method of residues by closing the contour in the lower half-plane and Eq. (16) can be substracted from the result to obtain

$$P_2(x,s) = -P_I(x,s) + \frac{1}{i}\sum_{m=-\infty}^{\infty} \frac{\exp\{-i[(\nu_m^- - Mk)x + (d^\dagger\Gamma_m - \nu_m^- s^\dagger)]\}(d^\dagger\Gamma_m - \nu_m^- s^\dagger)\kappa^-(\nu_m^-)}{d^\dagger(s^\dagger\Gamma_m - d^\dagger\nu_m^-)} \left\{ \frac{1}{(\nu_m^- - Mk)\kappa^-(Mk)} \right.$$
$$\left. \times\left[\tilde B_1 + i\tilde B_2\left(\frac{1}{\nu_m^- - Mk} - \frac{[\kappa^-(Mk)]'}{\kappa^-(Mk)}\right)\right] + \sum_{n=0}^{\infty}\left[\frac{S_n^+}{(\nu_m^- - \lambda_n)\kappa^-(\lambda_n)} + \frac{S_n^-}{(\nu_m^- + \lambda_n)\kappa^-(-\lambda_n)}\right]\right\} \quad \text{for } 0<x<s^\dagger \qquad (24)$$

IV. Numerical Results

Since $P=P_I+P_2$ and P_2 is identically zero in the region upstream of the trailing-edge Mach wave, the surface pressures can easily be calculated from Eqs. (14-16, and 24). The total lift and moment acting on the blade is obtained in the usual way by integrating these results (or their moments) over the surface of the blades. The dimensionless moment coefficient

$$\mathfrak{M} \equiv \int_{-s^\dagger}^{1-s^\dagger} (x-d_0)[P]\mathrm{d}x/A_0$$

is tabulated as a function of interblade phase angle for pure torsional motion about the center of each blade in Table 1. The compressible reduced frequency parameter k [defined by Eq. (4)] for these results is equal to 1, while the geometry corresponds to Verdon's cascade A.[5] This cascade has been used by a number of authors to establish the accuracy of their solution procedure. Verdon's computed results for this cascade (taken from Fig. 7 of his paper) appear in the last two columns in the table. To within the accuracy with which the figure could be read, Verdon's results are identical to those computed from the present analysis except in the vicinity of resonance [redacted] where Verdon's integral formulation fails to converge. The reason his formulation encounters computational difficulties near resonance, while the present formulation does not, will be discussed later in this section.

Figures 5-10 illustrate the effects of various cascade geometry parameters (i.e., solidity, stagger angle), inlet Mach number, and cascade dynamic response (i.e., reduced frequency, pitching axis location, interblade phase angle) on the unsteady aerodynamic moment for torsional motion. In all cases the interblade phase angle σ is taken as a parameter along the curves. For torsional motion, the work per cycle (done by the flow on the blades) is equal to $\pi A_0 \rho U_\infty^2 \mathrm{Im}\,\mathfrak{M}$.[12] When this quantity is positive, the blades receive energy from the flow and become unstable when this energy is greater than the (usually small) mechanical energy dissipated by the system. Hence, the cascade will tend to flutter when Im $(\mathfrak{M}/A_0)$ is positive. Figures 5-7 show that at constant inlet Mach number increasing the reduced frequency has a stabilizing influence. From these figures it also appears that for Mach numbers greater than 1.4 the range of interblade phase angle over which the imaginary part of the moment coefficient is positive is a weak function of the inlet Mach number. The effects of cascade solidity and stagger angle are illustrated in Figs. 8 and 9, respectively. Figure 8 shows that increasing the cascade solidity has a slight destabilizing effect on the cascade. Similarly Fig. 9 shows that at constant inlet Mach number the stability of the cascade can be increased by reducing the stagger angle.

Figure 10 shows the effect of pitching axis location on the aerodynamic damping. For the case shown the rearward movement of the pitching axis does cause a radical change in the shape of the response curves. Futhermore, these results indicate that the magnitude of the unstable region is strongly influenced by the pitching axis location. This result and the results shown on previous figures suggest that reduced frequency and pitching axis location are the most important parameters for controlling the onset of supersonic unstalled torsional flutter.

A cursory investigation of the equations for the surface pressure distribution would indicate that there may be factors in the denominator of Eqs. (14-16, and 24) that go to zero at certain operating conditions and thereby cause the blade forces to become infinite. However, a closer examination

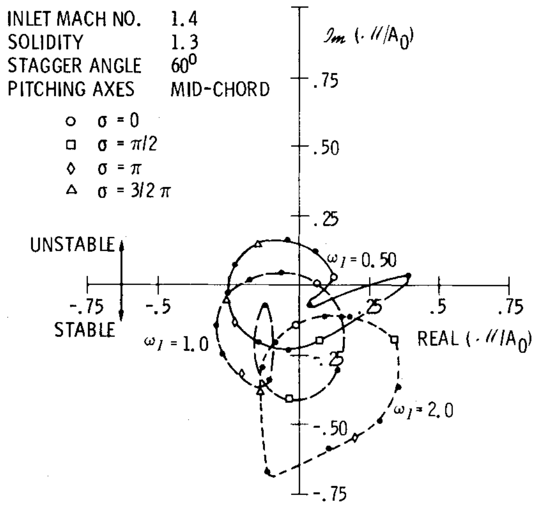

Fig. 6 Complex moment coefficient for pitching motion about center airfoil. Black dots denote intermediate values (i.e., multiples of $\pi/6$) of interblade phase angle σ. Mach number 1.4.

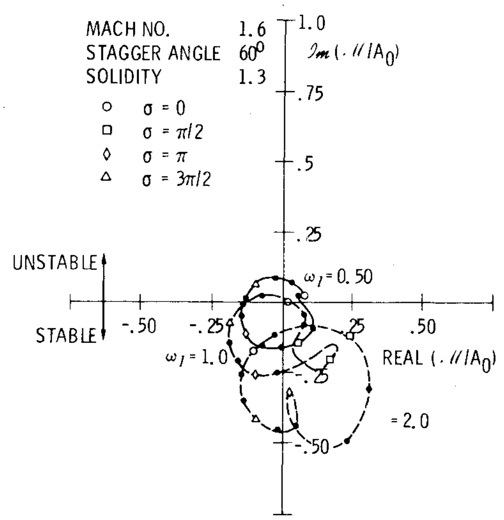

Fig. 7 Complex moment coefficient for pitching motion about center of airfoil. Black dots denote intermediate values (i.e., multiples of $\pi/6$) of interblade phase angle σ. Mach number 1.6.

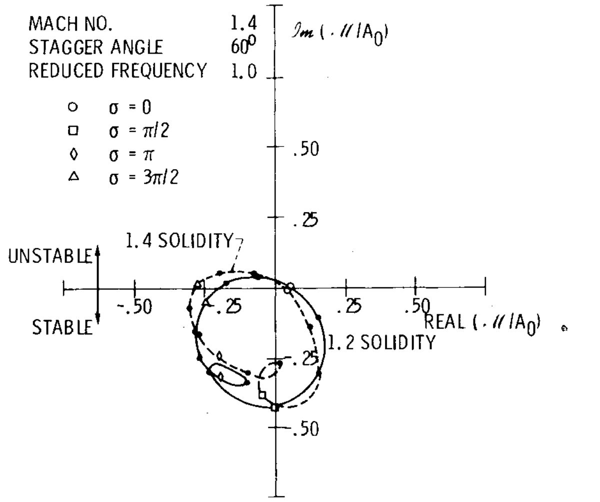

Fig. 8 The effects of cascade solidity on the complex moment coefficient for pitching motion about center of airfoil.

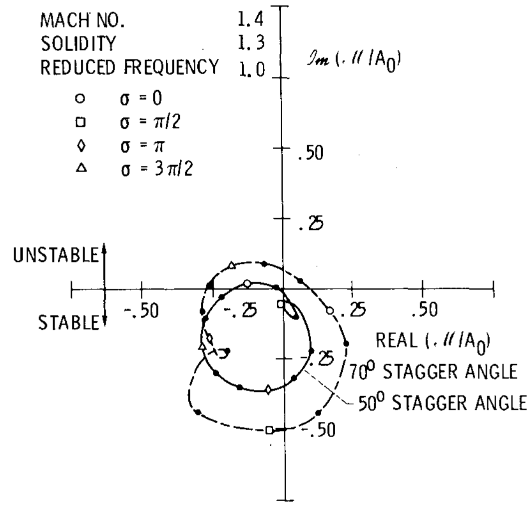

Fig. 9 The effect of cascade stagger angle on the complex moment coefficient for pitching motion about center of airfoil.

reveals that when zeros of the denominators do occur, they are balanced by corresponding zeros in the numerator.

It should be noted that the infinite duct resonant condition plays no special role in the present problem.[14] Previous investigators of the supersonic cascade problem who used numerical or semianalytical techniques[4,5] have raised a question about whether the blade forces vanish at the infinite duct resonant condition, but the present result, which is exact, shows that no such behavior will occur. This apparent discrepancy between the present results and those of previous investigators can easily be resolved by referring to the analysis developed in Ref. 3. Previous studies which have exhibited resonance behavior involve the solution of an integral equation of the form [Eq. (13) of Ref. 3]

$$v(x,0) = \int_{-l/2}^{l/2} K_0(x-x')[P(x')]\mathrm{d}x'$$

for the aerodynamic loading $[P(x')]$. At the infinite duct resonant condition the kernel function K_0 becomes singular. This singularity is independent of $(x-x')$ and is caused by the vanishing of the factor $s^\dagger\Gamma_n - d^{\dagger 2}\alpha_n^\pm$ that appears in the denominator of K_0 [i.e., see Eqs. (19) and (20)]. In order for the upwash velocity to remain finite the zero in the denominator of the kernel function must be balanced by a corresponding zero in the numerator. From Eqs. (19) and (20) of Ref. 3 it is seen that the only term that can vanish in the numerator at resonance is the integral

$$\int_{-l/2}^{l/2} [P(x')]\exp[-i(\alpha_n^+ - Mk)x']\mathrm{d}x' \qquad (25)$$

In the past it has been assumed that the vanishing of this integral implied the vanishing of $[P(x')]$ at every point of $-1/2<x'<1/2$. But the integral can also vanish as a result of cancellations between the positive and negative portions of the integrand. Calculations based on the present analysis show that the loading $\int_{-l/2}^{l/2} [P(x')]\mathrm{d}x'$ remains finite at resonance and hence that the latter behavior must occur. This has been confirmed by using the surface pressure equations derived in this paper to evaluate the integral in Eq. (25).

Typical results of this evaluation are plotted as a function of interblade phase angle in Fig. 11. The cascade geometry is Verdon's cascade A which exhibited resonance at an interblade phase angle of 85.38 deg. Figure 11 clearly shows that the integral in Eq. (25) vanishes at the infinite duct resonant condition while $\int_{-l/2}^{l/2} [P(x')\mathrm{d}x'$ does not. Hence, unlike the subsonic cascade where the aerodynamic loading (i.e., Ref. 13) approaches zero at resonance, the supersonic cascade blade forces exhibit no special behavior at this condition.

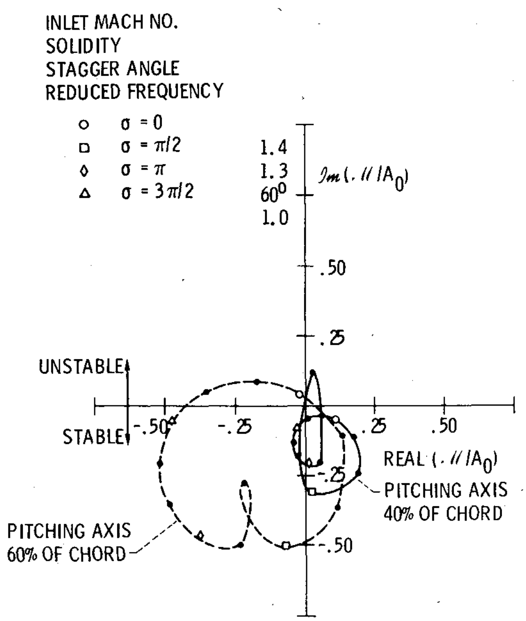

Fig. 10 The effect of pitching axes on the complex moment coefficient.

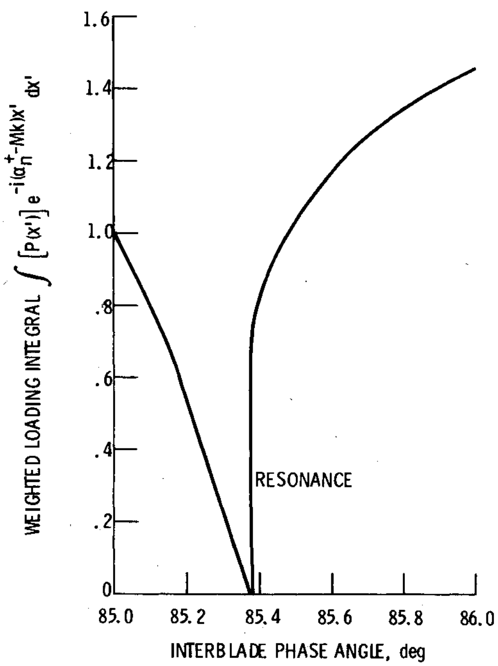

Fig. 11 The behavior of integral in Eq. (25) near duct resonance operating conditions.

V. Concluding Remarks

An exact analytical solution for the unsteady supersonic cascade problem has been obtained. Although the results obtained from the present formulation agree with those found previously by numerical or semianalytical methods the present closed-form solution has a number of advantages. First, the closed-form structure of the solution allows for very rapid and accurate calculations of the aerodynamic force, moment, and surface pressure distribution. Second, various limiting solutions can easily be extracted from the present formulation. These solutions include results for a freestream Mach number of unity. Third, the solution technique can be extended (by means of local linearization methods or Oswatitsch approximation) to incorporate finite loading effects. Finally, the analysis demonstrates that no resonance behavior will occur at supersonic velocities.

References

[1]Kurosaka, M., "On the Unsteady Supersonic Cascade with a Subsonic Leading Edge—An Exact First Order Theory—Part 1," *Journal of Engineering for Power, Transactions of ASME*, Ser. A, Vol. 96, Jan. 1974, pp. 13-22.

[2]Verdon, J. M. and McCune, J. E., "The Unsteady Supersonic Cascade in Subsonic Axial Flow," AIAA Paper 75-22, Jan. 1975.

[3]Goldstein, M. E., "Cascade with Subsonic Leading-Edge Locus," *AIAA Journal*, Vol. 13, Aug. 1975, pp. 1117-1119; see also "On the Kernel Function for the Unsteady Supersonic Cascade with Subsonic Leading Edge Locus," NASA TM X-71673, Feb. 1975.

[4]Nagashima, T. and Whitehead, D. S., "Aerodynamic Forces and Moments for Vibrating Supersonic Cascade Blades," University of Cambridge, Dept. of Engineering, Rept. CUED/A-Turbo/TR 59, 1974.

[5]Verdon, J. M., "Further Developments in the Aerodynamic Analysis of Unsteady Supersonic Cascades Parts 1 and 2," *Journal of Engineering for Power, Transactions of ASME*, Ser. A, Vol. 99, Oct. 1977, pp. 509-525.

[6]Verdon, J. M., "The Unsteady Aerodynamics of a Finite Supersonic Cascade with Subsonic Axial Flow," *Journal of Applied Mechanics*, Vol. 40, Sept. 1973, pp. 667-671.

[7]Brix, C. W. and Platzer, M. F., "Theoretical Investigation of Supersonic Flow Past Oscillating Cascades with Subsonic Leading-Edge Locus," AIAA Paper 74-14, Jan. 1974.

[8]Goldstein, M. E., Braun, W., and Adamczyk, J. J., "Unsteady Flow in a Supersonic Cascade with Strong In-Passage Shocks," *Journal of Fluid Mechanics*, Vol. 83, Pt. III, Dec. 1977, pp. 569-605.

[9]Noble, B. *Methods Based on the Wiener-Hopf Technique*, Pergamon Press, New York, 1958.

[10]Mani, R. and Horvay, G., "Sound Transmission Through Blade Rows," *Journal of Sound and Vibration*, Vol. 12, 1970, pp. 59-83.

[11]Lane, F., "System Mode Shapes in the Flutter of Compressor Blade Rows," *Journal of the Aeronautical Sciences*, Vol. 23, Jan. 1956, pp. 54-66.

[12]Fung, Y. C., *An Introduction to the Theory of Aeroelasticity*, Wiley, New York, 1955, pp. 166-168.

[13]Lane, F. and Friedman, M., "Theoretical Investigation of Subsonic Oscillatory Blade-Row Aerodynamics," NACA TR No. 4136, 1958.

[14]Kurosaka, M., "On the Issue of Resonance in an Unsteady Supersonic Cascade," *AIAA Journal*, Vol. 13, Nov. 1975, p. 1514.